

Optimal tie-breaking rules*

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Abstract

We consider two-player contests with the possibility of ties and study the effect of different tie-breaking rules on effort. For ratio-form and difference-form contests that admit pure-strategy Nash equilibrium, we find that the effort of both players is monotone decreasing in the probability that ties are broken in favor of the stronger player. Thus, the effort-maximizing tie-breaking rule commits to breaking ties in favor of the weaker agent. With symmetric agents, we find that the equilibrium is generally symmetric and independent of the tie-breaking rule. We also study the design of random tie-breaking rules that are ex-ante fair and identify sufficient conditions under which breaking ties before the contest actually leads to greater expected effort than the more commonly observed practice of breaking ties after the contest.

1 Introduction

Contests are situations in which agents exert costly effort to win one or more prizes. Examples of such competitive situations include sporting contests, promotional tournaments, political contests, R&D races, etc. In many of these situations, it is often the case that there is no outright winner and the contest ends in a draw or a tie. Moreover, a draw may not be an acceptable outcome for the designer. For instance, in sports competitions such as cricket, chess, and soccer, a significant fraction of the games end in a draw. But if these games happen to be knockout games of a world event, the designer must determine a single winner. Many different tie-breaking rules have been used to determine a winner in such situations. The result might be decided by chance¹ or it might be pre-determined based on

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¹The outcome of many elections that ended with ties have been determined via a coin toss (e.x. Kentucky city mayor race 2022). Even in sporting competitions where ties are broken by another short duration contest, like the super over in cricket or the penalty shootout in soccer, it can be argued that the outcome is almost a random draw as there is relatively little scope for skill or effort to have an impact.

some personal attributes like age, sex, height, weight, or status (incumbent or challenger)². In this paper, we consider contests between two agents and focus on understanding how different tie-breaking rules compare in terms of the effort they induce.

We study the effect of tie-breaking rules for three contest environments that differ in how the effort exerted by the players determine the distribution over contest outcomes (player 1 wins, player 2 wins, or tie). We study ratio-form contests (Tullock [35], Baik [2], Ewerhart [14], Wang [37], Nti [31, 32]), difference-form contests (Hirshleifer [19], Baik [1], Skaperdas [33], Beviá and Corchón [4], Che and Gale [7]), and lastly, we discuss examples of concave contests (Blavatsky [5], Fu and Wu [16]). While the literature has typically assumed only two possible outcomes³, we will consider generalizations of these contests that allow for the possibility of ties (Blavatsky [5], Jia [22], Vesperoni and Yildizparlak [36]). A feature of some of these generalizations is that the probability of tie increases as the efforts come closer, and in particular, it is maximized when both agents exert equal efforts. Motivated by this, with ratio-form and difference-form contests, we assume that the probability of a tie increases as the contest becomes more equal (ratio of efforts goes to 1 or difference goes to 0). In addition, we make assumptions that ensure existence and uniqueness of pure-strategy Nash equilibrium.

We make two primary contributions. First, we find that when the two agents differ in their value from winning, their effort is decreasing in the probability that the ties are broken in favor of the stronger player. Thus, an effort-maximizing contest designer would prefer to commit to breaking ties in favor of the weaker player. In this way, our results lend support to the idea of leveling the playing field and increasing the competitive balance of a contest to increase effort. Second, we find that with symmetric agents, the equilibrium is generally symmetric and independent of the tie-breaking rule. We also discuss the design of random unbiased tie-breaking rules and identify conditions under which breaking ties before the contest would lead to greater effort than the standard practice of breaking them after the contest has ended in a tie. For all our results, we identify parametric classes of contests that satisfy our assumptions and discuss the application of our results to them.

There is a growing literature studying the effect of introducing ties on equilibrium effort in contests. For concave contests with ties, introduced by Loury [25] and axiomatized by Blavatsky [5], the possibility of a tie has been shown to reduce total equilibrium effort (Nti [30], Deng et al. [11], Li et al. [24]) though it may increase winner's expected effort (Deng et al. [11]) or even total effort in a winner-pay setting (Minchuk [28]). In all-pay auctions, finite strategy spaces and bid gaps have been used to study the effect of introducing ties

²In weightlifting contests, ties were resolved, until 2016, in favor of the lighter athlete. In boxing, if a championship bout ends in a draw, the champion usually retains the title. In the 1999 Cricket World Cup, the tied semi-final match between Australia and South Africa went to Australia because they had defeated South Africa earlier in the tournament.

³Surveys of this literature can be found in Garfinkel and Skaperdas [17], Jia et al. [23], Corchón and Serena [10], Chowdhury et al. [8], Mealem and Nitzan [27]

(Eden et al. [13], Cohen and Sela [9], Gelder et al. [18]). Other related work has illustrated the merits of introducing draws in different contests (Nalebuff and Stiglitz [29], Imhof and Kräkel [20, 21], Chang et al. [6]). In comparison to this literature, a tie is a natural outcome in our contests and the prize is awarded irrespective of the contest outcome.

The paper contributes to the literature studying the effect of draw prizes on effort. Most of the work in this domain assumes symmetric agents and studies the effect of a common draw prize on effort. For a generalization of ratio-form contests to allow for ties, Vesperoni and Yildizparlak [36] find that there is a unique symmetric equilibrium that does not depend on the common draw prize⁴. In ratio-form contests, Jia [22] shows that there is always a unique equilibrium that is symmetric even though the contest might be biased⁵. They illustrate how introducing ties with no prizes can lead to equilibrium that is no longer symmetric. In comparison to this literature, our model allows for asymmetric agents and can be interpreted as studying the effects of awarding complementary draw prizes. We find that for both Vesperoni and Yildizparlak [36] and Jia [22] contests, the equilibrium effort is decreasing in the probability that a tied contest is awarded to the stronger player. In a similar spirit, Szech [34] finds that an asymmetric tie-breaking rule that favors the weaker player increases expected bids in an all-pay auction environment.

The paper also contributes to the literature studying the effect of leveling the playing field on effort in contests with heterogeneous agents. There are many different mechanisms that have been studied including multiplicative biases and additive headstarts⁶. In short, it has been found that if the designer cares about increasing effort and can optimally choose multiplicative biases, it cannot gain from being able to further choose additive biases or introduce draws (Fu and Wu [16], Franke et al. [15], Li et al. [24]). We believe ours is the first to consider this idea of leveling the playing field in the context of tie-breaking rules. We note here that for some contests (like the class of concave contests from Fu and Wu [16]), the choice of a tie-breaking rule is essentially equivalent to defining different head-starts for the two players. However, the tie-breaking rule imposes constraints on the set of feasible head-starts (such as the sum of head-starts must be a constant) and so the problem of finding the optimal tie-breaking rule in these contests is still different from the problem of finding optimal headstarts as considered in these papers.

The paper proceeds as follows. In section 2, we present our model of a two-player contest with ties. In section 3 and 4, we study ratio-form and difference-form contests respectively. In section 5, we discuss some other contest examples and study the design of random unbiased tie-breaking rules. Section 6 concludes. Most of the proofs are relegated to the appendix.

⁴They also find conditions under which the equilibrium effort is greater with the possibility of draw as compared to without it, even though the draw prize may be 0.

⁵This is referred to as the homogeneity paradox as one would probably expect that players put in different levels of effort if the contest favors one player over another.

⁶Surveys of this literature can be found in Mealem and Nitzan [27], Chowdhury et al. [8].

2 Model

There are two risk-neutral players competing in a contest. The contest has three possible outcomes: player 1 wins, player 2 wins, or it may be a tie. The distribution over the three outcomes is determined by the efforts $x_1 \geq 0, x_2 \geq 0$ exerted by the two players. We let $p_i(x_1, x_2)$ denote the probability that player i wins and $p_0(x_1, x_2)$ denote the probability that the contest ends in a tie. Thus, for any $x_1, x_2 \geq 0$, $p_1 + p_2 + p_0 = 1$. In case of a tie, the designer awards the contest to player 1 with probability $q \in [0, 1]$ and awards it to player 2 with remaining probability $1 - q$. We refer to q as the *tie-breaking rule*. Note that q is independent of the effort exerted by the agents. Given the tie-breaking rule q , player 1 eventually wins the contest with probability $P_1 = p_1 + qp_0$ while player 2 eventually wins with probability $P_2 = p_2 + (1 - q)p_0$. The value of player $i \in \{1, 2\}$ from winning the contest is V_i , where we assume $V_1 \geq V_2 > 0$. The cost of exerting effort x_i for player i is given by $c(x_i)$ where $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a cost function. We will assume either linear $c(x) = x$ or quadratic costs $c(x) = x^2$ in our analysis.

Given a contest $C = (V_1, V_2, p_1(), p_2(), q, c())$, player 1's payoff under profile (x_1, x_2) is

$$\Pi_1(x_1, x_2) = V_1(p_1(x_1, x_2) + qp_0(x_1, x_2)) - c(x_1),$$

and that of player 2 is

$$\Pi_2(x_1, x_2) = V_2(p_2(x_1, x_2) + (1 - q)p_0(x_1, x_2)) - c(x_2).$$

An effort profile (x_1^*, x_2^*) is a pure-strategy Nash equilibrium if

$$\Pi_1(x_1^*, x_2^*) \geq \Pi_1(x_1, x_2^*) \text{ for all } x_1 \in \mathbb{R}_+ \text{ and } \Pi_2(x_1^*, x_2^*) \geq \Pi_2(x_1^*, x_2) \text{ for all } x_2 \in \mathbb{R}_+.$$

We will impose conditions on C that guarantee the existence and uniqueness of a pure-strategy Nash equilibrium, characterized by the first-order conditions. We note here the two first-order conditions. The first-order condition for player 1 is

$$\frac{\partial \Pi_1}{\partial x_1} = 0 \implies V_1 \left(\frac{\partial p_1}{\partial x_1} + q \frac{\partial p_0}{\partial x_1} \right) = c'(x_1), \quad (1)$$

and that for player 2 is

$$\frac{\partial \Pi_2}{\partial x_2} = 0 \implies V_2 \left(\frac{\partial p_2}{\partial x_2} + (1 - q) \frac{\partial p_0}{\partial x_2} \right) = c'(x_2). \quad (2)$$

Given $V_1, V_2, p_1(), p_2(), c()$, we consider the designer's problem of choosing a tie-breaking rule $q \in [0, 1]$ so as to maximize the total effort in the pure-strategy Nash equilibrium⁷. Formally, the designer's problem is

$$\max_{q \in [0, 1]} x_1(q) + x_2(q)$$

where $x_i(q)$ refers to the equilibrium effort level of player i under the contest given by $C = (V_1, V_2, p_1(), p_2(), q, c())$. Going forward, we will study and solve the designer's problem for different classes of contest games.

⁷We note here that for some of our results, we only need the objective to be an increasing function of the efforts.

3 Ratio-form contest success functions

In this section, we consider instances of our model where the distribution over the three outcomes depends only on the ratio of the efforts $\theta = \frac{x_1}{x_2}$ of the two players. Formally, we assume that there exist twice-differentiable functions $p : \mathbb{R}_+ \rightarrow [0, 1]$ and $p_0 : \mathbb{R}_+ \rightarrow [0, 1]$ such that

- $p_1(x_1, x_2) = p(\theta)$ and $p_2(x_1, x_2) = p(\frac{1}{\theta})$,
- $p_0(x_1, x_2) = p_0(\theta)$ and $p_0(\theta) = p_0(\frac{1}{\theta})$ for all $\theta \in \mathbb{R}_+$,
- $c(x) = x$.

We'll make the following assumption on these functions.

Assumption 1. Let $z_q(\theta) = p(\theta) + qp_0(\theta)$.

- For any $q \in [0, 1]$, $z'_q(\theta) > 0$, $z''_q(\theta) < 0$, and $2z'_q(\theta) + \theta z''_q(\theta) > 0$, for all θ in \mathbb{R}_+ .
- For any $q \in [0, 1]$, $z_q(0) = 0$ and $\lim_{\theta \rightarrow \infty} z_q(\theta) = 1$

Note that $z_q(\theta)$ denotes the probability that player 1 eventually wins the prize and $1 - z_q(\theta)$ is the probability that player 2 eventually wins the prize. The first part of the assumption then says that a player's probability of winning is increasing in its own effort, decreasing in the effort of the other player, and in addition, it is increasing at a decreasing rate in one's own effort. The second part of the assumption says that if a player exerts 0 effort while the other player exerts a positive effort level, its probability of winning is 0 irrespective of the tie-breaking rule q .

The next assumption we make is on the probability of ties $p_0(\theta)$ and is motivated by the idea that the probability of a tie increases as the contest becomes more closely contested.

Assumption 2. The probability of tie $p_0(\theta)$ is increasing for $\theta \in (0, 1]$ and decreasing for $\theta \in [1, \infty)$.

We first provide a characterization of the pure-strategy Nash equilibria of the ratio-form contest game with ties.

Theorem 1. *Consider a ratio-form contest satisfying Assumption 1. The contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 \beta z'_q(\beta) \text{ and } x_2^* = V_2 \beta z'_q(\beta),$$

where $\beta = \frac{V_1}{V_2} \geq 1$.

To prove the result, we show that there is a unique solution to the first-order conditions 1 and 2 for the contest game defined by x_1^* and x_2^* . The concavity of the payoffs implied by Assumption 1 means that the second-order conditions for maximization are satisfied⁸. We note here that the result also follows from the result of Baik [2] who characterized the pure-strategy Nash equilibrium of two player ratio-form contests satisfying Assumption 1.

The next result shows that the effort-maximizing tie-breaking rule breaks ties in favor of the weaker agent.

Theorem 2. *Consider a ratio-form contest satisfying Assumptions 1 and 2.*

- If $V_1 > V_2$, $x_1^*(q) + x_2^*(q)$ is decreasing in q and so the optimal tie-breaking rule is $q = 0$.
- If $V_1 = V_2$, $x_1^*(q) + x_2^*(q)$ is independent of q .

First consider the case where the agents are symmetric so that $\beta = \frac{V_1}{V_2} = 1$. Observe that in this case, both agents exert equal effort, irrespective of the bias introduced by the tie-breaking rule q . Intuitively, this is because at any symmetric profile, a player's effort has zero marginal impact on the probability of a tie, and therefore, the tie-breaking rule q does not affect the symmetric equilibrium. With asymmetric agents, we again have that the stronger agent puts in greater effort, irrespective of the bias due to the tie-breaking rule q . In particular, the marginal impact of q on player i 's effort is simply $V_i \beta p'_0(\beta)$. Since $\beta > 1$, $p'_0(\beta) < 0$ and it follows that the effort of both agents goes down as q increases. Thus, even if the designer had a more general objective function that was increasing in the effort of both agents, it would still be optimal for the designer to bias the tie-breaking rule completely in favor of the weaker player with $q = 0$. Formally, the result follows from the fact that the total equilibrium effort is linear in q with the coefficient $(V_1 + V_2)\beta p'_0(\beta)$. The full proof is in the appendix.

The Tullock contests, defined by $p_i(x_i, x_{-i}) = \frac{x_i^r}{x_i^r + x_{-i}^r}$ for $r \in (0, \infty)$, is an important class of ratio-form contest success functions that have been widely studied in the literature. There has been significant work in generalizing the Tullock contests to allow for the possibility of ties. We will consider three such generalizations. For two of these generalizations, the contest success functions take the ratio form and we discuss how our results apply to them next. The third generalization by Blavatsky [5] is discussed later.

The first generalization we consider is given by $p_i(x_i, x_j) = \frac{x_i^{rk}}{(x_i^r + x_j^r)^k}$ with $k \geq 1$ so that the probability of ties is $p_0(x_i, x_j) = \frac{(x_i^r + x_j^r)^k - x_i^{rk} - x_j^{rk}}{(x_i^r + x_j^r)^k}$. This was proposed by Vesperoni and Yildizparlak [36]. In this case, we obtain the following lemma.

⁸Note that Assumption 1 implies that the players payoffs Π_i are strictly concave in their actions x_i . Since the argument relies on there being a unique solution to the first-order conditions, we can relax Assumption 1 so that payoffs are only required to be quasiconcave and the result would go through.

Lemma 1. *Suppose C is a ratio-form contest with $p_i(x_i, x_j) = \frac{x_i^{rk}}{(x_i^r + x_j^r)^k}$ where $k \geq 1$. If $rk \leq 1$, then for any $q \in [0, 1]$, the contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 rk \frac{\beta^{rk} + q(\beta^r - \beta^{rk})}{(1 + \beta^r)^{k+1}} \text{ and } x_2^* = V_2 rk \frac{\beta^{rk} + q(\beta^r - \beta^{rk})}{(1 + \beta^r)^{k+1}},$$

where $\beta = \frac{V_1}{V_2}$.

To prove the result, we define the functions $p(\theta), p_0(\theta)$ and show that they satisfy Assumptions 1, 2 when $rk \leq 1$ with $r \leq 1, k \geq 1$. Note that the conditions in the lemma are sufficient but not necessary for the result. The full proof is in the appendix.

The second generalization we consider is given by $p_i(x_i, x_j) = \frac{x_i^r}{x_i^r + kx_j^r}$ with $k \geq 1$ so that the probability of ties is $p_0(x_i, x_j) = 1 - \frac{x_i^r}{x_i^r + kx_j^r} - \frac{x_j^r}{kx_i^r + x_j^r}$. This was proposed by Jia [22]. In this case, we obtain the following lemma.

Lemma 2. *Suppose C is a ratio-form contest with $p_i(x_i, x_j) = \frac{x_i^r}{x_i^r + kx_j^r}$ where $k \geq 1$. If $r \leq 1$, then for any $q \in [0, 1]$, the contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 rk \beta^r \left(\frac{1}{(\beta^r + k)^2} + q \left(\frac{1}{(1 + k\beta^r)^2} - \frac{1}{(\beta^r + k)^2} \right) \right)$$

and

$$x_2^* = V_2 rk \beta^r \left(\frac{1}{(\beta^r + k)^2} + q \left(\frac{1}{(1 + k\beta^r)^2} - \frac{1}{(\beta^r + k)^2} \right) \right),$$

where $\beta = \frac{V_1}{V_2}$.

Again, we define the functions $p(\theta), p_0(\theta)$ and show that Assumptions 1 and 2 are satisfied when $r \leq 1$. The full proof is in the appendix. Observe that in Lemmas 1 and 2, if we plug in $k = 1$, we get back the equilibrium characterization for the Tullock contests without ties (Nti [31]). Also, the total effort under the optimal tie-breaking rule ($q = 0$) in the Vesperoni and Yildizparlak [36] contest is $(V_1 + V_2)rk \frac{\beta^{rk}}{(1 + \beta^r)^{k+1}}$ and that under the Jia [22] contest is $(V_1 + V_2)rk \frac{\beta^r}{(k + \beta^r)^2}$. Since both are increasing in k at $k = 1$, it follows that if the designer could induce ties by increasing k and then commit to breaking them in favor of the weaker player, it would induce greater effort than in the standard Tullock contests.

4 Difference-form contest success functions

In this section, we consider instances of our model where the distribution over the three outcomes depends only on the difference of the efforts $\theta = x_1 - x_2$ of the two players. Formally, we assume that there exist twice-differentiable functions $p : \mathbb{R} \rightarrow [0, 1]$ and $p_0 : \mathbb{R} \rightarrow [0, 1]$ such that

- $p_1(x_1, x_2) = p(\theta)$ and $p_2(x_1, x_2) = p(-\theta)$,
- $p_0(x_1, x_2) = p_0(\theta)$ and $p_0(\theta) = p_0(-\theta)$ for all $\theta \in \mathbb{R}$,
- $c(x) = x^2/2$.

We'll make the following assumption on these functions.

Assumption 3. Let $z_q(\theta) = p(\theta) + qp_0(\theta)$. For any $q \in [0, 1]$, $z'_q(\theta) > 0$ and $z''_q(\theta) \in [-\frac{1}{V_1}, \frac{1}{V_1}]$.

Here again, $z_q(\theta)$ denotes the probability that player 1 eventually wins the prize and $1 - z_q(\theta)$ is the probability that player 2 eventually wins the prize. The assumption then says that a player's probability of winning is increasing in its own effort, decreasing in the effort of the other player. The second assumption ensures that the player's objective function is globally concave and a unique best response exists.

The next assumption we make is on the probability of ties $p_0(\theta)$ and is again motivated by the idea that the probability of a tie increases as the contest becomes more closely contested.

Assumption 4. The probability of tie $p_0(\theta)$ is increasing for $\theta \in (-\infty, 0]$ and decreasing for $\theta \in [0, \infty)$.

Let us now characterize the pure-strategy Nash equilibria of this difference-form contest game with ties.

Theorem 3. Consider a difference-form contest satisfying Assumption 3. The contest has a unique pure-strategy Nash equilibrium defined by

$$x_1^* = V_1 z'_q(\beta(q)) \text{ and } x_2^* = V_2 z'_q(\beta(q)),$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2)z'_q(\theta)$.

To prove the result, we show that there is a unique solution to the first-order conditions 1 and 2 for the contest game defined by x_1^* and x_2^* . The concavity of the payoffs implied by Assumption 1 means that the second-order conditions for maximization are satisfied⁹. The full proof is in the appendix.

Note that the solution $\beta(q)$ to the equation $\theta = (V_1 - V_2)z'_q(\theta)$ is unique and ≥ 0 for all $q \in [0, 1]$. It represents the difference in equilibrium effort levels exerted by the two agents. That is, the equilibrium effort levels x_1^*, x_2^* are such that $\frac{x_1^*}{x_2^*} = \frac{V_1}{V_2}$ and $x_1^* - x_2^* = \beta(q)$.

Next, we discuss the effect of the tie-breaking rule q on total equilibrium effort.

Theorem 4. Consider a difference-form contest satisfying Assumptions 3 and 4.

- If $V_1 > V_2$, $x_1^*(q) + x_2^*(q)$ is decreasing in q and so the optimal tie-breaking rule is $q = 0$.
- If $V_1 = V_2$, $x_1^*(q) + x_2^*(q)$ is independent of q .

⁹Assumption 3 implies that the players payoffs Π_i are concave in x_i . The result goes through if the payoffs Π_i are instead assumed to be quasiconcave in x_i .

To prove Theorem 4, we can use Theorem 3 and our assumptions on p_0 to see how the total effort changes as we increase q . When $V_1 > V_2$, player 1 puts in greater effort so that $p'_0(x_1^* - x_2^*) < 0$. This implies that $R(q)$ is decreasing in q and thus, the optimal tie-breaking rule sets $q = 0$. In comparison, when $V_1 = V_2$, the agents exert equal effort irrespective of q and because $p'_0(0) = 0$, it follows that the total effort does not depend on the choice of q . The full proof is in the appendix.

The difference-form contest success functions were first studied in Hirshleifer [19]. A well-known example, with zero probability of ties, is the logit function $p_i(x_1, x_2) = \frac{\exp(x_i)}{\exp(x_i) + \exp(x_{-i})}$. While we are not aware of explicit work on generalizing these contest success functions to allow for the possibility of ties, we consider a couple of generalizations inspired from the literature on generalizing ratio-form contest success functions to allow for the possibility of ties.

The first generalization we consider is given by $p_i(x_i, x_j) = \left(\frac{\exp(x_i)}{\exp(x_i) + \exp(x_{-i})} \right)^k$ with $k \geq 1$. In this case, we obtain the following lemma.

Lemma 3. *Suppose C is a difference-form contest with $p_i(x_i, x_j) = \left(\frac{\exp(x_i)}{\exp(x_i) + \exp(x_{-i})} \right)^k$ with $k \geq 1$. There exist some $\bar{V} > 0$ such that if $\bar{V} \geq V_1 \geq V_2 > 0$, then for any $q \in [0, 1]$, the contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 z'_q(\beta(q)) \text{ and } x_2^* = V_2 z'_q(\beta(q)),$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2) z'_q(\theta)$.

To prove the result, we show that $z'_q(\theta) > 0$ and also $\lim_{\theta \rightarrow \infty} z'_q(\theta) = 0$ and $\lim_{\theta \rightarrow -\infty} z''_q(\theta) = 0$ which implies $z''_q(\theta)$ is bounded between some M and m . Thus, if we take V_1 small enough so that $\frac{-1}{V_1} \leq m \leq M \leq \frac{1}{V_1}$, Assumption 3 will be satisfied. We further show that this contest satisfies Assumption 4 so that Theorem 4 applies.

The next generalization we consider is given by $p_i(x_i, x_j) = \frac{\exp(x_i)}{\exp(x_i) + k \exp(x_{-i})}$ with $k \geq 1$. In this case, we obtain the following lemma.

Lemma 4. *Suppose C is a difference-form contest with $p_i(x_i, x_j) = \frac{\exp(x_i)}{\exp(x_i) + k \exp(x_{-i})}$ with $k \geq 1$. There exist some $\bar{V} > 0$ such that if $\bar{V} \geq V_1 \geq V_2 > 0$, then for any $q \in [0, 1]$, the contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 z'_q(\beta(q)) \text{ and } x_2^* = V_2 z'_q(\beta(q)),$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2) z'_q(\theta)$.

As before, we show that $z'_q(\theta) > 0$ and also $\lim_{\theta \rightarrow \infty} z'_q(\theta) = 0$ and $\lim_{\theta \rightarrow -\infty} z''_q(\theta) = 0$ which implies $z''_q(\theta)$ is bounded between some M and m . It follows that 3 and 4 will be satisfied if V_1 is small enough and thus, we can apply Theorems 3 and 4 to get the result.

5 Discussion

In this section, we consider some other examples of contests and also discuss the design of random tie-breaking rules.

5.1 Concave contests

Here we consider contests $C = (V_1, V_2, p_1(), p_2(), q, c())$ that satisfy the following assumption.

Assumption 5. The contest success functions take the form

$$p_i(x_1, x_2) = \frac{f_i(x_i)}{f_1(x_1) + f_2(x_2) + 1} \text{ and } p_0(x_1, x_2) = \frac{1}{f_1(x_1) + f_2(x_2) + 1}$$

where $f_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is twice-differentiable, strictly increasing, and concave.

This class of contest success functions with a possibility of tie was axiomatized by Blavatsky [5]. For such contest success functions, we have the following lemma.

Lemma 5. *Suppose the contest C satisfies Assumption 5. Then, for any increasing and convex cost function c with $c(0) = 0$ and any tie-breaking rule $q \in [0, 1]$, the contest admits a unique pure-strategy Nash equilibrium.*

Given a tie-breaking rule $q \in [0, 1]$, we can define impact functions $g_1(x_1) = f_1(x_1) + q$ and $g_2(x_2) = f_2(x_2) + (1 - q)$ which, together with a convex cost function, constitute a regular concave contest (as defined in Definition 1 of Fu and Wu [16]). The result then follows from Theorem 1 of Fu and Wu [16] which says that there exists a unique pure-strategy Nash equilibrium in regular concave contest games¹⁰.

The unique pure-strategy Nash equilibrium may involve both players exerting positive effort, in which case it is characterized by the first-order conditions 1 and 2. Going forward, we will focus on the special case where $f_1(x) = f_2(x) = f(x) = x^r$ for $r \in (0, 1]$ and $c(x) = x$. With these parametric assumptions, the first-order conditions are given by

$$V_1 P_2 r x_1^{r-1} = x_1^r + x_2^r + 1 = V_2 P_1 r x_2^{r-1},$$

where $P_2 = p_2 + (1 - q)p_0$ and $P_1 = p_1 + qp_0$.

We note here that for $r = 1$, the solution to these conditions is

$$x_1^* = \frac{V_1^2 V_2}{(V_1 + V_2)^2} - q \text{ and } x_2^* = \frac{V_1 V_2^2}{(V_1 + V_2)^2} - (1 - q),$$

and it will constitute the unique pure-strategy Nash equilibrium as long as $x_1^*, x_2^* \geq 0$. Observe that the headstarts defined by the tie-breaking rule q act as a direct substitute for the effort exerted by them and the total equilibrium effort is independent of the choice of q . The next lemma illustrates how the tie-breaking rule matters in the case where $r = 0.5$.

¹⁰Observe that a tie-breaking rule essentially defines additive head-starts q and $1 - q$ in the Blavatsky [5] model. While previous work has studied the problem of finding optimal head-starts (Franke et al. [15], Fu and Wu [16], Li et al. [24]), our problem of finding the optimal tie-breaking rule is different in that the head-starts must add up to some constant.

Lemma 6. Suppose the contest C is such that $p_i(x_1, x_2) = \frac{\sqrt{x_i}}{\sqrt{x_i} + \sqrt{x_j} + 1}$, $c(x) = x$ and $V_1 = V_2 = V$.

- For any tie-breaking rule $q \in [0, 1]$, the contest has a unique pure-strategy Nash equilibrium defined by

$$x_1^* = \frac{1}{4(2V + 1)} \left(V - (\sqrt{2V + 1} - 1)q \right)^2$$

and

$$x_2^* = \frac{1}{4(2V + 1)} \left(V - (\sqrt{2V + 1} - 1)(1 - q) \right)^2.$$

- The total effort $x_1^* + x_2^*$ is convex in $q \in [0, 1]$ with a minimum at $q = 0.5$ and maximum at $q = 0$ and $q = 1$.

Even though the agents are identical, a designer who cares about maximizing effort is better off biasing the contest in favor of one of the agents by committing to breaking ties in its favor. Note that this is in contrast to ratio-form and difference-form contests where we saw that with identical agents, the choice of tie-breaking rule has no effect on effort.

5.2 Random tie-breaking rules

As we have discussed, there are many situations where a designer would want to bias the contest in favor of one of the players, even when they are ex-ante identical (as in Lemma 6). But designing such a biased contest may be controversial or even infeasible if the designer is not aware of the relative strengths of the two players. In such situations, the designer may still be able to introduce bias into the contest while being fair ex-ante. For example, instead of arbitrarily committing to breaking ties in favor of one of the players, it can publicly toss a fair coin before the contest begins to pre-determine a winner in case the contest ends in a tie¹¹. This corresponds to a *random tie-breaking rule* in which the tie-breaking rule q is 0 or 1 with equal probability. Note that this is different from the tie-breaking rule $q = 0.5$ in which case a fair coin is tossed to determine the winner after the contest has ended in a tie.

More generally, a *random tie-breaking rule* is defined by a random variable Q with support in $[0, 1]$ and we say that the rule is *unbiased* if $\mathbb{E}[Q] = 0.5$. The timing of the contest with random tie-breaking rules is as follows:

1. The contest designer chooses a distribution for random variable Q with support in $[0, 1]$. This is the random tie-breaking rule and is known to all participants.
2. The value of Q is realized and publicly revealed. This is the tie-breaking rule $q \in [0, 1]$.

¹¹Many sports have coin tosses before the contest begins which may introduce some bias. For example, in cricket, a coin is tossed before the match begins and the winner of the toss gets to choose whether they want to bat or bowl first. There is some evidence that winning the toss provides a small, but significant improvement to a team's chances of winning.

3. Agents decide their effort levels.
4. The contest outcome is revealed and in case of a tie, player 1 is chosen as winner with probability q and player 2 is chosen as winner with probability $1 - q$.

Next, we will discuss the design of random unbiased tie-breaking rules. More precisely, we will consider the problem of choosing a random tie-breaking rule Q to maximize expected total effort $\mathbb{E}[x_1(Q) + x_2(Q)]$ under the constraint that the rule is unbiased so that $\mathbb{E}[Q] = 0.5$.

First, for ratio-form contests, we obtain the following as a corollary of Theorem 1.

Corollary 1. *In a ratio-form contest satisfying Assumptions 1 and 2, any random unbiased tie-breaking rule leads to the same expected total effort.*

This follows from the fact that the total equilibrium effort is linear in q .

For the difference-form contests, we identify a sufficient condition under which breaking ties by tossing a fair coin before the contest would be optimal.

Lemma 7. *Consider a difference-form contest satisfying Assumptions 3 and 4. If $V_1 > V_2$ and $p_0''(\theta) < 0$ for all $\theta \in [0, \sqrt{2V_1}]$, the total effort is convex in q and the optimal unbiased random tie-breaking rule chooses $Q = 0$ and $Q = 1$ with equal probability.*

We note that when V_1 is small enough, the assumption $p_0''(\theta) < 0$ for all $\theta \in [0, \sqrt{2V_1}]$ is satisfied for the difference-form contest success functions considered in Lemmas 3 and 4.

Finally, for concave contests with square-root impact function and identical agents, it follows from Lemma 6 that breaking ties in a fair way before the contest maximizes effort among all random unbiased tie-breaking rules. Thus, a designer may be better off pre-determining the winner in case of a tie by tossing a fair coin before the contest begins as compared to the traditional practice of tossing a fair coin after the contest ends.

6 Conclusion

We study two-player contests with the possibility of ties under both ratio-form and difference-form contest success functions. In these contests, we study the effect different tie-breaking rules have on the effort exerted by the players. When players are heterogeneous, we find that the total effort decreases as the probability that ties are broken in favor of the stronger agent increases. Thus, an effort-maximizing designer would prefer to commit to breaking ties in favor of the weaker agent. The result lends further support to the encouraging effect of leveling the playing field on effort in contests with heterogeneous agents.

With symmetric agents, we find that the equilibrium is generally symmetric and does not depend on the choice of tie-breaking rule in the case of ratio-form and difference-form contests. The problem is more interesting for concave contests in which we make some parametric assumptions and find that an effort-maximizing designer would prefer to pre-determine the winner in case of a tie by tossing a fair coin before the contest begins as compared to the standard practice of breaking ties after the contest ends. We believe that the study of tie-breaking rules for concave contests and contests with more than two agents provide interesting directions for future research.

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A Proofs for Section 3 (Ratio-form contest success functions)

Theorem 1. *Consider a ratio-form contest satisfying Assumption 1. The contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 \beta z'_q(\beta) \text{ and } x_2^* = V_2 \beta z'_q(\beta),$$

where $\beta = \frac{V_1}{V_2} \geq 1$.

Proof. In a ratio-form contest $C = (V_1, V_2, p_1(), p_2(), q, c())$, there exist functions p and p_0 so that we can rewrite players payoff under profile x_1, x_2 as

$$\Pi_1 = V_1 z_q(\theta) - x_1 \text{ and } \Pi_2 = V_2(1 - z_q(\theta)) - x_2$$

where $\theta = \frac{x_1}{x_2}$ and $z_q(\theta) = p(\theta) + qp_0(\theta)$.

Since the prize is valued positively by both players and the probability of winning the contest with $x_i = 0, x_j > 0$ is 0 for player i and 1 for player j , any pure-strategy Nash equilibrium involves positive effort levels by both players.

Now, observe that

$$\frac{\partial \Pi_1}{\partial x_1} = \frac{V_1}{x_2} z'_q(\theta) - 1 \text{ and } \frac{\partial \Pi_2}{\partial x_2} = \frac{x_1 V_2}{x_2^2} z'_q(\theta) - 1$$

and

$$\frac{\partial^2 \Pi_1}{\partial x_1^2} = \frac{V_1}{x_2^2} (z''_q(\theta)) < 0 \text{ and } \frac{\partial^2 \Pi_2}{\partial x_2^2} = -\frac{x_1 V_2}{x_2^3} (2z'_q(\theta) + \theta z''_q(\theta)) < 0.$$

From Assumption 1, Π_i is concave in x_i for any x_j . Since any pure-strategy Nash equilibrium must satisfy the first-order conditions

$$V_1 z'_q(\theta) = x_2 \text{ and } x_1 V_2 z'_q(\theta) = x_2^2,$$

concavity of payoffs implies that the second-order conditions are satisfied at any solution to the first-order conditions. Observe that any solution to the two first-order conditions must satisfy

$$\frac{x_1}{x_2} = \frac{V_1}{V_2} = \beta.$$

But this condition uniquely pins down the solution to the first-order conditions:

$$x_1^* = V_1 \beta z'_q(\beta) \text{ and } x_2^* = V_2 \beta z'_q(\beta).$$

□

Theorem 2. *Consider a ratio-form contest satisfying Assumptions 1 and 2.*

- If $V_1 > V_2$, $x_1^*(q) + x_2^*(q)$ is decreasing in q and so the optimal tie-breaking rule is $q = 0$.
- If $V_1 = V_2$, $x_1^*(q) + x_2^*(q)$ is independent of q .

Proof. From Theorem 1, the total effort in equilibrium is given by

$$R(q) = (V_1 + V_2)\beta(p'(\beta) + qp'_0(\beta)).$$

Observe that the total effort is linear in q with the coefficient $(V_1 + V_2)\beta p'_0(\beta)$ and the sign of the coefficient is determined by $p'_0(\beta)$. When $\beta = \frac{V_1}{V_2} > 1$ Assumption 2 implies that $p'_0(\beta) < 0$ and it follows that the total effort is decreasing in q . So the optimal tie-breaking rule breaks ties in favor of the weaker player ($q = 0$).

Similarly, when $\beta = \frac{V_1}{V_2} = 1$ Assumption 2 implies that $p'_0(\beta) = 0$ and it follows that the equilibrium effort, and thus, the total effort, is independent of q . So the choice of the tie-breaking rule does not matter in this case. \square

Lemma 1. *Suppose C is a ratio-form contest with $p_i(x_i, x_j) = \frac{x_i^{rk}}{(x_i + x_j)^k}$ where $k \geq 1$. If $rk \leq 1$, then for any $q \in [0, 1]$, the contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 rk \frac{\beta^{rk} + q(\beta^r - \beta^{rk})}{(1 + \beta^r)^{k+1}} \text{ and } x_2^* = V_2 rk \frac{\beta^{rk} + q(\beta^r - \beta^{rk})}{(1 + \beta^r)^{k+1}},$$

where $\beta = \frac{V_1}{V_2}$.

Proof. To prove the result, we define the functions $p(\theta), p_0(\theta)$ and show that they satisfy Assumptions 1, 2 when $rk \leq 1$ with $r \leq 1, k \geq 1$.

For the given contest, we can define the functions

$$p(\theta) = \frac{\theta^{rk}}{(1 + \theta^r)^k}, \quad p_0(\theta) = \frac{(1 + \theta^r)^k - 1 - \theta^{rk}}{(1 + \theta^r)^k}$$

so that

$$z_q(\theta) = \frac{\theta^{rk} + q((1 + \theta^r)^k - 1 - \theta^{rk})}{(1 + \theta^r)^k}.$$

Let us first identify conditions under which Assumption 1 is satisfied.

To do so, we have that

$$\begin{aligned} z'_q(\theta) &= \frac{(1 + \theta^r)^k (rk\theta^{rk-1}(1 - q) + qkr(1 + \theta^r)^{k-1}\theta^{r-1}) - (\theta^{rk} + q((1 + \theta^r)^k - 1 - \theta^{rk})) (kr(1 + \theta^r)^{k-1}\theta^{r-1})}{(1 + \theta^r)^{2k}} \\ &= kr\theta^{r-1} \frac{(\theta^{rk-r}(1 - q)) + q}{(1 + \theta^r)^{k+1}} \\ &= kr \frac{\theta^{rk-1}(1 - q) + q\theta^{r-1}}{(1 + \theta^r)^{k+1}}. \end{aligned}$$

Also,

$$\begin{aligned} z''_q(\theta) &= kr\theta^{r-2} \frac{q(r-1) + \theta^{rk-r}(rk-1)(1-q) - \theta^{rk}(1+r)(1-q) - \theta^r q(1+kr)}{(1+\theta^r)^{k+2}} \\ &= kr\theta^{r-2} \frac{q(r-1 - \theta^r(1+kr)) + (1-q)\theta^{rk-r}(rk-1 - \theta^r(1+r))}{(1+\theta^r)^{k+2}}. \end{aligned}$$

Lastly,

$$2z'_q(\theta) + \theta z''_q(\theta) = kr\theta^{r-1} \frac{q(r+1 + \theta^r(1-kr)) + (1-q)\theta^{rk-r}(rk+1 + \theta^r(1-r))}{(1+\theta^r)^{k+2}}.$$

From the above expressions, we have that $z'_q(\theta) > 0$ for all $\theta > 0$. The condition $rk \leq 1$ with $r \leq 1$ and $k \geq 1$ is sufficient for both $z''_q(\theta) < 0$ and $2z'_q(\theta) + \theta z''_q(\theta) > 0$. Thus, we have that if $rk \leq 1$, Assumption 1 is satisfied.

Observe that

$$p'_0(\theta) = kr \frac{\theta^r - \theta^{rk}}{\theta(1+\theta^r)^{k+1}},$$

which is ≥ 0 for $\theta \leq 1$ and ≤ 0 for $\theta \geq 1$ since $k \geq 1$. Thus, Assumption 2 is also satisfied. \square

Lemma 2. *Suppose C is a ratio-form contest with $p_i(x_i, x_j) = \frac{x_i^r}{x_i^r + kx_j^r}$ where $k \geq 1$. If $r \leq 1$, then for any $q \in [0, 1]$, the contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 rk\beta^r \left(\frac{1}{(\beta^r + k)^2} + q \left(\frac{1}{(1+k\beta^r)^2} - \frac{1}{(\beta^r + k)^2} \right) \right)$$

and

$$x_2^* = V_2 rk\beta^r \left(\frac{1}{(\beta^r + k)^2} + q \left(\frac{1}{(1+k\beta^r)^2} - \frac{1}{(\beta^r + k)^2} \right) \right),$$

where $\beta = \frac{V_1}{V_2}$.

Proof. To prove the result, we define the functions $p(\theta), p_0(\theta)$ and show that they satisfy Assumptions 1, 2 when $r \leq 1$. We further show that the condition in Theorem 2 is also satisfied when $r \leq 1$ which implies the result.

For the given contest, we can define the functions

$$p(\theta) = \frac{\theta^r}{\theta^r + k}, \quad p_0(\theta) = 1 - \frac{\theta^r}{\theta^r + k} - \frac{1}{k\theta^r + 1}$$

so that

$$z_q(\theta) = \frac{\theta^r(1 + k\theta^r + q(k^2 - 1))}{(1 + k\theta^r)(\theta^r + k)}.$$

Let us first identify conditions under which Assumption 1 is satisfied. To do so, we have that

$$z'_q(\theta) = kr\theta^{r-1} \frac{(1-q)(1+k\theta^r)^2 + q(\theta^r + k)^2}{(1+k\theta^r)^2(\theta^r + k)^2}.$$

Also,

$$z''_q(\theta) = -kr\theta^{r-2} \left(2r\theta^r \left(\frac{1-q}{(\theta^r + k)^3} + \frac{qk}{(1+k\theta^2)^3} \right) + (1-r) \left(\frac{1-q}{(\theta^r + k)^2} + \frac{q}{(1+k\theta^2)^2} \right) \right).$$

Lastly,

$$2z'_q(\theta) + \theta z''_q(\theta) = kr\theta^{r-1} \left(\frac{(1-q)}{(\theta^r + k)^2} \left(1 + r - \frac{2r\theta^r}{\theta^r + k} \right) + \frac{q}{(1+k\theta^r)^2} \left(1 + r - \frac{2rk\theta^r}{1+k\theta^r} \right) \right).$$

From the above expressions, we have that $z'_q(\theta) > 0$ for all $\theta > 0$. The condition $r \leq 1$ is sufficient for both $z''_q(\theta) < 0$ and $2z'_q(\theta) + \theta z''_q(\theta) > 0$. Thus, we have that if $r \leq 1$, Assumption 1 is satisfied.

Observe that

$$\begin{aligned} p'_0(\theta) &= kr\theta^{r-1} \frac{(\theta^r + k)^2 - (1+k\theta^r)^2}{(1+k\theta^r)^2(\theta^r + k)^2} \\ &= kr\theta^{r-1} \frac{(\theta^r + k + 1 + k\theta^r)(\theta^r + k - 1 - k\theta^r)}{(1+k\theta^r)^2(\theta^r + k)^2} \\ &= kr\theta^{r-1} \frac{(\theta^r + k + 1 + k\theta^r)(k-1)(1-\theta^r)}{(1+k\theta^r)^2(\theta^r + k)^2}, \end{aligned}$$

which is ≥ 0 for $\theta \leq 1$ and ≤ 0 for $\theta \geq 1$ since $k \geq 1$. Thus, Assumption 2 is also satisfied. \square

B Proofs for Section 4 (Difference-form contest success functions)

Theorem 3. *Consider a difference-form contest satisfying Assumption 3. The contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 z'_q(\beta(q)) \text{ and } x_2^* = V_2 z'_q(\beta(q)),$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2) z'_q(\theta)$.

Proof. In a difference-form contest $C = (V_1, V_2, p_1(), p_2(), q, c())$, there exist functions p and p_0 so that we can rewrite players payoff under profile x_1, x_2 as

$$\Pi_1 = V_1 z_q(\theta) - \frac{x_1^2}{2} \text{ and } \Pi_2 = V_2(1 - z_q(\theta)) - \frac{x_2^2}{2}$$

where $\theta = x_1 - x_2$ and $z_q(\theta) = p(\theta) + qp_0(\theta)$.

Now, observe that

$$\frac{\partial \Pi_1}{\partial x_1} = V_1 z'_q(\theta) - x_1 \text{ and } \frac{\partial \Pi_2}{\partial x_2} = V_2 z'_q(\theta) - x_2$$

and

$$\frac{\partial^2 \Pi_1}{\partial x_1^2} = V_1(z''_q(\theta)) - 1 < 0 \text{ and } \frac{\partial^2 \Pi_2}{\partial x_2^2} = -V_2(z''_q(\theta)) - 1 < 0.$$

From Assumption 3, Π_i is concave in x_i for any x_j . Since any pure-strategy Nash equilibrium must satisfy the first-order conditions

$$V_1 z'_q(\theta) = x_1 \text{ and } V_2 z'_q(\theta) = x_2,$$

concavity of payoffs implies that the second-order conditions for maximization are satisfied at any solution to the first-order conditions. Now any solution to the two first-order conditions must satisfy the following:

$$\frac{x_1}{x_2} = \frac{V_1}{V_2}$$

and

$$x_1 - x_2 = (V_1 - V_2)z'_q(x_1 - x_2).$$

Consider the equation $\theta - (V_1 - V_2)z'_q(\theta) = 0$. The derivative of the left hand side is $1 - (V_1 - V_2)z''_q(\theta)$ which is ≥ 0 as long as $z''_q(\theta) \leq \frac{1}{V_1 - V_2}$. From Assumption 3, $z''_q(\theta) \leq \frac{1}{V_1}$ which implies $z''_q(\theta) \leq \frac{1}{V_1 - V_2}$. Thus, the left hand side is monotone increasing in θ . Also observe that at $\theta = 0$, the left hand side is $-(V_1 - V_2)z'_q(0) < 0$. Thus, there is a unique solution to the equation $\theta - (V_1 - V_2)z'_q(\theta) = 0$. Let $\beta(q)$ denote this unique solution to the equation $\theta - (V_1 - V_2)z'_q(\theta) = 0$. The uniqueness of $\beta(q)$ implies that there is a unique solution to the first-order conditions and it takes the form

$$x_1^* = \frac{\beta(q)V_1}{V_1 - V_2} \text{ and } x_2^* = \frac{\beta(q)V_2}{V_1 - V_2}.$$

Equivalently, we can write it as

$$x_1^* = V_1 z'_q(\beta(q)) \text{ and } x_2^* = V_2 z'_q(\beta(q)).$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2)z'_q(\theta)$. □

Theorem 4. Consider a difference-form contest satisfying Assumptions 3 and 4.

- If $V_1 > V_2$, $x_1^*(q) + x_2^*(q)$ is decreasing in q and so the optimal tie-breaking rule is $q = 0$.
- If $V_1 = V_2$, $x_1^*(q) + x_2^*(q)$ is independent of q .

Proof. From Theorem 3, the total effort in the unique pure-strategy Nash equilibrium is given by

$$R(q) = (V_1 + V_2)z'_q(\beta(q)),$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2)z'_q(\theta)$.

Consider the case where $V_1 > V_2$. In this case, we have that

$$R(q) = (V_1 + V_2)z'_q(\beta(q)) = \frac{V_1 + V_2}{V_1 - V_2}\beta(q)$$

from the equation defining $\beta(q)$. Thus, $R'(q) = \frac{V_1 + V_2}{V_1 - V_2}\beta'(q)$ which implies that the total equilibrium effort $R(q)$ goes in the same direction as the difference in equilibrium effort $\beta(q)$ as we change q . From the characterizing equation, we know that

$$\beta'(q) = (V_1 - V_2)(p''(\beta(q))\beta'(q) + p'_0(\beta(q)) + qp''_0(\beta(q))\beta'(q)),$$

which implies

$$\beta'(q)(1 - (V_1 - V_2)(p''(\beta(q)) + qp''_0(\beta(q)))) = (V_1 - V_2)p'_0(\beta(q)).$$

Given that $z''_q(\theta) \in [-\frac{1}{V_1}, \frac{1}{V_1}]$ from Assumption 3, we know that $\beta'(q)$ has the same sign as $p'_0(\beta(q))$. We already know $\beta(q) > 0$ and therefore, from Assumption 4, we get that $\beta'(q) < 0$. Therefore, the total equilibrium effort $R(q)$ is decreasing in q and it follows that the optimal tie breaking rule breaks ties in favor of the weaker agent by setting $q = 0$.

When $V_1 = V_2$, $\beta(q) = 0$ for all $q \in [0, 1]$ and thus, $R(q) = (V_1 + V_2)(p'(0) + qp'_0(0))$. But we know from Assumption 4 that $p'_0(0) = 0$ and thus, we get that the total effort equals $R(q) = (V_1 + V_2)p'(0)$ for any $q \in [0, 1]$. Thus, with symmetric agents, the choice of the tie-breaking rule does not matter for equilibrium, and thus, total effort. \square

Lemma 3. Suppose C is a difference-form contest with $p_i(x_i, x_j) = \left(\frac{\exp(x_i)}{\exp(x_i) + \exp(x_{-i})}\right)^k$ with $k \geq 1$. There exist some $\bar{V} > 0$ such that if $\bar{V} \geq V_1 \geq V_2 > 0$, then for any $q \in [0, 1]$, the contest has a unique pure-strategy Nash equilibrium defined by

$$x_1^* = V_1 z'_q(\beta(q)) \text{ and } x_2^* = V_2 z'_q(\beta(q)),$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2)z'_q(\theta)$.

Proof. To prove the result, we define the functions $p(\theta), p_0(\theta)$ and show that they satisfy Assumptions 3, 4 if V_1 is small enough. The result then follows from Theorems 3 and 4.

For the given contest, we can define the functions

$$p(\theta) = \frac{e^{k\theta}}{(1+e^\theta)^k}, \quad p_0(\theta) = \frac{(1+e^\theta)^k - 1 - e^{k\theta}}{(1+e^\theta)^k}$$

so that

$$z_q(\theta) = (1-q) \frac{e^{k\theta}}{(1+e^\theta)^k} + q \left(1 - \frac{1}{(1+e^\theta)^k} \right).$$

Let us first identify conditions under which Assumption 3 is satisfied.

To do so, we have that

$$z'_q(\theta) = \frac{ke^\theta}{(1+e^\theta)^{k+1}} \left((1-q)e^{(k-1)\theta} + q \right).$$

Also,

$$z''_q(\theta) = \frac{ke^\theta}{(1+e^\theta)^{k+2}} \left((1-q)e^{(k-1)\theta}(k-e^\theta) + q(1-ke^\theta) \right).$$

Observe that $z'_q(\theta) > 0$. Letting $t = e^\theta$, we get

$$z''_q(\theta) = \frac{kt}{(1+t)^{k+2}} \left((1-q)t^{(k-1)}(k-t) + q(1-kt) \right).$$

Since $\lim_{t \rightarrow \infty} z''_q(\theta) = 0$ and $\lim_{t \rightarrow 0} z''_q(\theta) = 0$, there exists some bounds m, M such that $m \leq z''_q(\theta) \leq M$ for all $\theta \in \mathbb{R}$. Thus, if we take V_1 small enough so that $\frac{-1}{V_1} \leq m \leq M \leq \frac{1}{V_1}$, Assumption 3 will be satisfied.

We also note that this contest success function satisfies Assumption 4 as

$$p'_0(\theta) = \frac{ke^\theta}{(1+e^\theta)^{k+1}} (1 - e^{(k-1)\theta}),$$

which is > 0 for $\theta < 0$ and < 0 for $\theta > 0$. Thus, Theorem 4 applies. \square

Lemma 4. *Suppose C is a difference-form contest with $p_i(x_i, x_j) = \frac{\exp(x_i)}{\exp(x_i) + k \exp(x_{-i})}$ with $k \geq 1$. There exist some $\bar{V} > 0$ such that if $\bar{V} \geq V_1 \geq V_2 > 0$, then for any $q \in [0, 1]$, the contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = V_1 z'_q(\beta(q)) \text{ and } x_2^* = V_2 z'_q(\beta(q)),$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2) z'_q(\theta)$.

Proof. To prove the result, we define the functions $p(\theta), p_0(\theta)$ and show that they satisfy Assumptions 3, 4 if V_1 is small enough. The result then follows from Theorems 3 and 4.

For the given contest, we can define the functions

$$p(\theta) = \frac{e^\theta}{k + e^\theta}, \quad p_0(\theta) = 1 - \frac{e^\theta}{k + e^\theta} - \frac{1}{ke^\theta + 1}$$

so that

$$z_q(\theta) = (1 - q) \frac{e^\theta}{k + e^\theta} + q \frac{ke^\theta}{1 + ke^\theta}.$$

Let us first identify conditions under which Assumption 3 is satisfied.

To do so, we have that

$$z'_q(\theta) = ke^\theta \left(\frac{1 - q}{(k + e^\theta)^2} + \frac{q}{(1 + ke^\theta)^2} \right).$$

Also,

$$z''_q(\theta) = ke^\theta \left(\frac{(1 - q)(k^2 - e^{2\theta})}{(k + e^\theta)^4} + \frac{q(1 - k^2e^{2\theta})}{(1 + ke^\theta)^4} \right).$$

Again, we have that $z'_q(\theta) > 0$ and also $\lim_{\theta \rightarrow \infty} z'_q(\theta) = 0$ and $\lim_{\theta \rightarrow -\infty} z'_q(\theta) = 0$ which implies $z''_q(\theta)$ is bounded between some M and m . Thus, if we take V_1 small enough so that $\frac{-1}{V_1} \leq m \leq M \leq \frac{1}{V_1}$, Assumption 3 will be satisfied.

We also note that this contest success function satisfies Assumption 4 as

$$p'_0(\theta) = ke^\theta \left(\frac{1}{(1 + ke^\theta)^2} - \frac{1}{(k + e^\theta)^2} \right),$$

which is > 0 for $\theta < 0$ and < 0 for $\theta > 0$. Thus, Theorem 4 applies. \square

C Proofs for Section 5 (Discussion)

Lemma 5. *Suppose the contest C satisfies Assumption 5. Then, for any increasing and convex cost function c with $c(0) = 0$ and any tie-breaking rule $q \in [0, 1]$, the contest admits a unique pure-strategy Nash equilibrium.*

Proof. Let f_1, f_2 denote the increasing and concave impact functions that define the contest success functions p_1, p_2, p_0 . For any $q \in [0, 1]$, we can write the probability that player i eventually wins the contest as

$$P_i = \frac{g_i(x_i)}{g_1(x_1) + g_2(x_2)},$$

where

$$g_1(x_1) = f_1(x_1) + qs \text{ and } g_2(x_2) = f_2(x_2) + (1 - q)s.$$

Observe that g_i is also twice-differentiable, strictly increasing and concave. Thus, it follows from Theorem 1 of Fu and Wu [16] that there exists a unique pure-strategy Nash equilibrium in this game. \square

Lemma 6. *Suppose the contest C is such that $p_i(x_1, x_2) = \frac{\sqrt{x_i}}{\sqrt{x_i} + \sqrt{x_j} + 1}$, $c(x) = x$ and $V_1 = V_2 = V$.*

- *For any tie-breaking rule $q \in [0, 1]$, the contest has a unique pure-strategy Nash equilibrium defined by*

$$x_1^* = \frac{1}{4(2V + 1)} \left(V - (\sqrt{2V + 1} - 1)q \right)^2$$

and

$$x_2^* = \frac{1}{4(2V + 1)} \left(V - (\sqrt{2V + 1} - 1)(1 - q) \right)^2.$$

- *The total effort $x_1^* + x_2^*$ is convex in $q \in [0, 1]$ with a minimum at $q = 0.5$ and maximum at $q = 0$ and $q = 1$.*

Proof. For the general case of $r < 1$ and $V_1 \geq V_2$, we get from the first-order conditions that the equilibrium effort levels x_1^* and x_2^* satisfies the equations

$$V_2 x_1 \left(1 + \frac{q}{x_1^r} \right) = V_1 x_2 \left(1 + \frac{1 - q}{x_2^r} \right)$$

and

$$V_2 r (x_1^r + q) = x_2^{1-r} (x_1^r + x_2^r + 1)^2.$$

In this special case where $r = 0.5$ with $V_1 = V_2 = V$, the equations simplify to:

$$\sqrt{x_1} (\sqrt{x_1} + q) = \sqrt{x_2} (\sqrt{x_2} + 1 - q) \quad V (\sqrt{x_1} + q) = 2\sqrt{x_2} (\sqrt{x_1} + \sqrt{x_2} + 1)^2$$

We can solve the two equations to get the unique pure strategy Nash equilibrium as in the lemma.

Let us verify that the x_1^* and x_2^* satisfy these two equations. To verify, let $a = V$ and $b = \sqrt{2V + 1}$ so that $x_1^* = \left(\frac{1}{2b}(a - (b - 1)q) \right)^2$ and $x_2^* = \left(\frac{1}{2b}(a - (b - 1)(1 - q)) \right)^2$. Then we have that

$$\sqrt{x_1^*} = \frac{1}{2b} (a - (b - 1)q) \quad \sqrt{x_1^*} + q = \frac{1}{2b} (a + (b + 1)q)$$

and

$$\sqrt{x_2^*} = \frac{1}{2b} (a - (b - 1)(1 - q)) \quad \sqrt{x_2^*} + 1 - q = \frac{1}{2b} (a + (b + 1)(1 - q)).$$

Using these, together with the fact that $2a = b^2 - 1$, we can verify that x_1^* and x_2^* solve the two first order conditions.

For the second part, observe that the total effort takes the form $x_1^* + x_2^* = (c - dq)^2 + (c - d(1 - q))^2$ for appropriately defined constants c and d that depend only on V . Upon simplifying, the total effort equals $2c^2 + d^2 - 2cd + 2d^2q(q - 1)$ which is clearly convex in q and has a minimum at $q = 0.5$ and maximum at $q = 0, 1$. \square

Lemma 7. *Consider a difference-form contest satisfying Assumptions 3 and 4. If $V_1 > V_2$ and $p_0''(\theta) < 0$ for all $\theta \in [0, \sqrt{2V_1}]$, the total effort is convex in q and the optimal unbiased random tie-breaking rule chooses $Q = 0$ and $Q = 1$ with equal probability.*

Proof. From Theorem 3, the total effort in the unique pure-strategy Nash equilibrium is given by

$$R(q) = (V_1 + V_2)z'_q(\beta(q)) = \frac{V_1 + V_2}{V_1 - V_2}\beta(q),$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2)z'_q(\theta)$.

Thus,

$$R''(q) = (V_1 + V_2)z''_q(\beta(q)) = \frac{V_1 + V_2}{V_1 - V_2}\beta''(q).$$

Note that the solution $\beta(q)$ of an implicit equation $F(\theta, q) = 0$ is convex iff

$$\frac{\partial F}{\partial q} \frac{\partial^2 F}{\partial \theta \partial q} - \frac{\partial F}{\partial \theta} \frac{\partial^2 F}{\partial q^2} \geq 0.$$

In our case, $\beta(q)$ is the solution to the equation $(V_1 - V_2)(p'(\theta) + qp_0'(\theta)) - \theta = 0$.

Then, $\beta(q)$ is convex if and only if

$$(V_1 - V_2)^2 p_0'(\theta) p_0''(\theta) \geq 0.$$

We know that for $V_1 > V_2$, $\theta > 0$ and so $p_0'(\theta) < 0$. Therefore, we need $p_0''(\theta) < 0$ for $\beta(q)$, and thus, $R(q)$ to be convex. Observe also that player 1's equilibrium effort x_1^* must be such that $V_1 - \frac{x_1^2}{2} \geq 0 \iff x_1^* \leq \sqrt{2V_1}$. Since $x_2^* \geq 0$, it follows that $\beta(q) \leq \sqrt{2V_1}$ for any q . Thus, if we have $p_0''(\theta) \leq 0$ for $\theta \in [0, \sqrt{2V_1}]$, we have that the total effort is convex in q . It follows from convexity then that the optimal random tie-breaking rule in the class of unbiased rules breaks ties in fair manner before the contest. \square